games

MDPI

Article

A Model of *k*-Winners

Diegο Armando Canales

Escuela de Ciencias Sociales y Gobierno, Tecnologico de Monterrey, Ave. Eugenio Garza Sada 2501 Sur, Colonia Tecnológico, Monterrey 64700, Mexico; diego.canales@tec.mx

Abstract: The concept of the Condorcet winner has become central to most electoral models in the political economy literature. A Condorcet winner is the alternative preferred by a plurality in every pairwise competition; the notion of a k-winner generalizes that of a Condorcet winner. The k-winner is the unique alternative top-ranked by the plurality in every competition comprising exactly k alternatives (including itself). This study uses a spatial voting setting to characterize this theoretical concept, showing that if a k-winner exists for some $k > 2$, then the same alternative must be the k'-winner for every $k' > k$. We derive additional results, including sufficient and necessary conditions for the existence of a k-winner for some $k > 2$.

Keywords: elections; k-winners; Condorcet winner

JEL Classification: D71; D72

1. Introduction

Designers of electoral procedures have long grappled with the challenge of accurately translating individual preferences into collective outcomes. Academics have widely studied the concept of the Condorcet winner, but relatively few have explored its generalization, which is the alternative preferred by a plurality in subsets of k alternatives: the k-winner. We argue that this concept can be of significant relevance in settings involving multi-stage elections with sequential eliminations.

In this paper, we present a framework that offers a simplified taxonomy using the concept of k-winners in a spatial voting context, where alternatives are ordered on a unit interval. More specifically, we provide necessary and sufficient conditions for the existence of a k-winner for some $k > 2$. In our setting, there must be a Condorcet winner, which is a 2-winner; however, this is not necessarily true for a k-winner when $k > 2$.

The first main result is Proposition 4, which suggests a monotonicity property: once an alternative becomes the majority choice from a size k subset of alternatives, it remains the majority choice for all larger group sizes ($k' > k$) within the electorate. This result implies that an alternative's vote share is robust and widespread enough to secure majority support even as the group size grows, offering insights into the stability of voting outcomes.

Furthermore, a particular alternative can be considered a stabilizing force across larger group sizes if it is the k-winner for some $k > 2$. This situation can mitigate the risk of cyclical outcomes where the subset of alternatives on a ballot diminishes over time, and the complex dynamics of primary elections in the US can exemplify this effect.

The primary mechanism in Propositions 5 and 6 is decisive, sufficiently large support across the electorate by the centrally positioned median alternative; Proposition 6 requires a weaker condition than Proposition 5.

Academic Editors: Ulrich Berger, Dimitri Landa and Le Bihan Patrick

Received: 24 May 2024
Revised: 19 December 2024
Accepted: 23 December 2024
Published: 1 February 2025

Citation: Canales, D. A. (2025). A Model of *k*-Winners. *Games*, *16*(1), 6. https://doi.org/10.3390/g16010006

The intuition behind the necessary and sufficient conditions in Proposition 6 hinges on the concept of a "decisive, sufficiently large vote share". The conditions in Proposition 6 specify the minimum vote share required for alternative a_m (i.e., $\frac{1}{2}(a_{m+1} - a_{m-1})$) to outperform alternatives a_{m+1} and a_{m-1} in a contest with exactly k alternatives, provided that ideal points for these alternatives are sufficiently distant from the ideal point of the median alternative.

Proposition 6 also includes an additional sufficient condition, requiring subsets of exactly k alternatives that are sufficiently small (see Lemmas 1 and 2). By jointly considering both Lemmas (i.e., $k \leqslant \min\{m, n-m\}$), the median alternative is the k-winner.

The intuition behind Propositions 7, 8, and 9 rests on a standard pattern regarding the subset size k: either k is larger than m or larger than $n - m$. In each case, we compute the vote share for alternative i required to surpass its competitors as the k-winner. This calculation identifies the "most damaging collection" of alternatives for alternative i regarding vote share: the two adjacent alternatives on either side and an external alternative (a_1 or a_n). The conditions presented in Propositions 7, 8, and 9 specify how far the alternative i must be from its competitors to become the k-winner.

Finally, the conditions in Propositions 10 and 11 are weaker than the sole condition in Proposition 3. Proposition 3 asserts that if a_1 or a_n has their ideal point closest to the median voter, at least half of the electorate would top-rank that alternative, making it the k-winner for any k. In contrast, Propositions 10 and 11 focus on the specific vote share required for alternative i to outperform the rest in a contest within a subset of k alternatives. These conditions precisely describe the minimum distance alternative i must maintain from its competitors to be the k-winner. In contrast, Proposition 3 does not explicitly state such distances. Section 5 presents several motivating examples where: (i) in a contest with three alternatives, both the 2-winner and the 3-winner exist but differ, and (ii) in a contest with four alternatives, the 2-winner and the 4-winner might differ, but importantly, the 3-winner does not exist. In this latter example, we can see that the number of remaining candidates determines the winner's identity, except when there are three remaining candidates; in this scenario, we have a three-way Condorcet cycle.

This paper is structured as follows: In Section 2, we will discuss a literature review of the concept of the Condorcet winner and its generalization. Section 3 introduces our model and completely characterizes k-winners in a spatial model. Section 4 contains the main results of this paper split into two subsections. Section 4.1 presents the sufficient and necessary conditions for the median alternative to be a k-winner for any $k > 2$. (ii) Sections 4.2 and 4.3 extend the previous results to non-median and exterior alternatives, respectively. Section 5 presents four numerical examples that illustrate the most relevant results. Finally, we present our concluding remarks in Section 6.

2. Literature Review

Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet Condorcet (1785), formalized the first and most cardinal discussion surrounding electoral phenomena. Condorcet's central contribution to voting theory is the concept of a Condorcet winner, defined as an alternative with the plurality of votes in a pairwise comparison against every other alternative. This paper presents a spatial approach that generalizes this notion into k-winners. A k-winner is an alternative with the plurality of votes in every k-wise comparison against every other alternative. Condorcet started looking at the fundamental problem a jury faces: collectively making a correct decision. His main arguments were based on the early notions of the computation of binomial probabilities (given two choices: truth and mistake in the original terms of the *Essai*). His intuition has been formalized as Condorcet's Jury Theorem. This result would constitute a gateway to a more general problem in the social sciences:

voting. Condorcet aimed to devise an optimal mathematical procedure to aggregate individual voting choices to select the optimal choice for the electorate. The idea was that each voter strictly ranks all the alternatives. The "winner" in a contest between some alternatives is the top-ranked by most voters. Condorcet considered a population of these alternatives and the contests between every pair of alternatives. The Marquis constructed the well-known "Condorcet criterion" by examining the alternative with the plurality of the votes in every head-to-head comparison. The Marquis suggested that this plurality alternative was society's "best" choice and should be elected; this became the definition of a "Condorcet winner". However, the Marquis discovered specific voting scenarios where voting yielded what he called "inconsistent propositions" (scenarios where the criterion could not compute a plurality winner). Moreover, inconsistent propositions dramatically rose as alternatives increased; in other words, Condorcet had discovered voting paradoxes. More than 170 years after the Marquis' seminal contribution, Black (1958) reintroduced, translated, and reinterpreted the central results from Condorcet's *Essai* in his 1958 book "The Theory of Committees and Elections". Black provided a remarkable workaround to Condorcet's paradox by restricting the class of voters' preferences and the population of alternatives. Black restricted voters' preferences to one most preferred outcome (their ideal point), and their preference for other outcomes decreases as they move away from this ideal point in either direction of the uni-dimensional policy space/spectrum. This restriction was the crucial element in the now-famous median voter theorem. As the alternatives are scalar, they can be ordered (i.e., $a_1 \leqslant \ldots \leqslant a_n$.); thus, we can order the voters by the alternatives that they top-rank. We define the median as the voter who top-ranks an alternative (say m); hence, at least half of voters top-rank some alternative that does not exceed m, and half of the voters top-rank an alternative that weakly exceeds m.

Theorem 1 (Black (1958))**.** *When n odd voters exist, all of whose utility functions are single-peaked on the real line, the ideal alternative of the median voter gets more votes in a contest against every other alternative, and it is the only alternative that can do so.*

Newing and Black (1951) later explored a Condorcet winner's (non)existence where the policy space is two-dimensional. Following his research program, Saari (1989) provided a theoretical framework where election outcomes could be categorized (and ordered) into a "dictionary". With a taxonomical description of voting systems, Saari then focused on the distinction where paradoxes might exist. He uses the concept of "binarily consistent", defined as when "the ranking for each subset of candidates is generated by the same complete, transitive, binary relationship". Therefore, intransitivity, such as the Condorcet paradox case, would be classified as "binarily inconsistent". Finally, Saari (1989) found that the set of "binarily inconsistent" words is a proper subset of the dictionary when using the Borda Count; according to Saari, "only Borda avoids many paradoxes". A related notion of k-winners is found in Meyers et al. (2014). The authors followed the Saari (1989) framework of the "dictionary" of voting outcomes and characterized "Condorcet words". The authors used this refinement of Saari's word to contextualize their concept of k-winners. The main contribution in Meyers et al. (2014) was characterizing the set of Condorcet words; in particular, they showed that some words are not Condorcet words. Their main characterizations of the existence of k-winners are highlighted in the following results. (i) Condorcet words exist whose first k elements are blanks in the first k positions and whose remaining elements are the same alternative. Additionally, (ii) Condorcet words exist whose first k elements are the same alternative and whose remaining elements are blanks. The present work differs in three critical respects. (i) Our main contribution is the sufficient and necessary conditions for the existence of a k-winner for any $k > 2$. (ii) This paper's definition of a k-winner is much less restrictive. (iii) We restrict the alternatives to be ordered on the unit interval. In particular, we fix a spatial profile and ask which, if any, alternative is a k-winner (in this work's sense). To illustrate the difference in concepts, an N-winner must always exist in our approach (irrespective of the spatial setting); however, as noted, there may be no N-winner in their sense.

3. Model

Let $a \in [0,1]$ be an alternative. A continuum of sincere-voting agents exists (i.e., voters), indexed by j, whose preferences over alternatives are represented by the loss function $-(a - a^j)^2$; a^j is voter j's ideal alternative in $[0,1]$.

3.1. Distribution of Ideal Alternatives

To simplify the exposition of our main results, we assume that a^j is uniformly distributed on $[0,1]$. The median voter is then the individual whose ideal alternative on $[0,1]$ is $1/2$. The assumption of uniformity, while admittedly restrictive, is employed solely to simplify the exposition of the results. Appendix A proposes an alternative formulation for the preliminaries of the model without the assumption of uniformity. We only assume that the distribution is continuous and has a unique median. We then reformulate Proposition 10 and the necessary and sufficient conditions for an a_m to be a k-winner for any given k.

3.2. Preliminaries

We focus on generic[1] collections of $n > 1$ (finite) alternatives (N) with representative element a_i. We order these alternatives such that

$$0 \leqslant a_1 < \dots < a_n \leqslant 1.$$

For any collection $K \subseteq N$, the lowest and highest alternatives in K are exterior to K, writing the former as $a_{\underline{K}}$ and the latter as $a_{\overline{K}}$. Any other alternatives are interior to K.

Each voter's loss function determines an ordering over the alternatives in any subset $K \subseteq N$. We denote $a^j(K)$ as the voter j's preferred alternative on K. We refer to $a^{1/2}(N)$, the median voter's preferred alternative in N, as the median alternative. Therefore, writing $a^{1/2}(N)$ as a_m is convenient. Take any nonempty subset of N, say K. For any $a \in K$, write $\theta(a;K)$ as the proportion of voters j such that $a = a^j(K)$.

The next assumption will be useful in removing the possibility of ties in any contest.

Assumption 1. *For every subset $K \in N$, no pair of alternatives can have the same vote share:*

$$\forall K : \theta(a;K) = \theta(a_{i'};K) \iff a = a_{i'}$$

Before presenting our definition of k-winners, we must define which alternative wins on any given subset K. a is a winner on K if $a \in \arg\max_{\alpha \in K} \theta(\alpha;K)$; a is a *loser on K* if $a \in \arg\min_{\alpha \in K} \theta(\alpha;K)$.

Next, we expand this notion to every collection of alternatives with k elements.

Definition 1. *Alternative a is a k-winner if alternative a is a winner on K for every nonempty $K \subseteq N$ with k elements; we define a k-loser analogously.*

Therefore, the 2-winner is a Condorcet winner, which coincides with $a^{1/2}(K)$, the median's preferred alternative for every pair of alternatives (K). If a k-winner exists, it must be unique, except for the trivial case of $k = 1$. If a_i is the only alternative available, a_i would win. This situation is true for every alternative.

Proposition 1 (Uniqueness). *For every $k \in \{2, \dots, n\}$, there is at most one k-winner.*

Proof. Fix any k and suppose by contradiction that two or more k-winners exist, say a_i and $a_{i'}$. Because N is generic, median voter m strictly prefers one of the pairs, say a_i. If $a_i < a_{i'}$, then a voter $j' > 1/2$ exists such that every voter $j < j'$ strictly prefers a_i over $a_{i'}$. At the same time, if $a_i > a_{i'}$ exists, then there is a voter $j' < 1/2$, such that every voter $j > j'$ strictly prefers a_i over $a_{i'}$. We next consider any collection of k alternatives (say, K) that contains a_i and $a_{i'}$. As $a_{i'}$ is a k-winner, it must be a winner on K; however, this is impossible because more than half of the voters prefer a_i over $a_{i'}$. Consequently, neither a_i nor $a_{i'}$ can be k-winners. □

The following proposition provides a set of sufficient conditions for the existence of the k-winner for every $k > 1$. Essentially, the alternative with the majority of the votes (i.e., more than one-half of the total votes) will be the strongest alternative to compete with (see the first example in Section 4.1). Furthermore, Proposition 2 exploits the fact that any voter who top-ranks a in a contest K must also top-rank a in every contest $K' \subset K$ such that $a \in K'$.

Proposition 2. *If the alternative a has $\theta(a; N) > \frac{1}{2}$ then it is the k-winner for every $k > 1$.*

Proof. Regardless of any combination of alternatives that run against alternative a_i, they would have the plurality of the votes and thus win every k-wise competition. □

Next, we present the first set of main results. The following two propositions explore the relationship between a k-winner (if it exists) and a $(k+1)$-winner. First, Proposition 3 establishes one sufficient condition for the 2-winner to be the 3-winner and the k-winner for every $k > 2$. Essentially, the following proposition aims to link the distinction between alternatives exterior to N and alternatives interior to N and the notion of a k-winner for every $k > 1$ in a spatial setting context.

Proposition 3. *If $a \in \{a_1, a_n\}$ is the 2-winner, then it is the k-winner for every $k > 1$.*

Proof. The median voter top ranks any 2-winner. If $a = a_1$, then there is $j^{'} \geqslant 1/2$, such that every voter $j < j^{'}$ strictly prefers a over every $a_i > a_1$. Hence, $\theta(a_1; K) \geqslant 1/2$ for every collection K that contains a_1; therefore, a_1 must be the k-winner. An equivalent argument applies if $a = a_n$ □

An inherent relationship exists in our spatial context between the k-winner and the $(k+1)$-winner for $k > 2$. The relevance of this result becomes apparent when we consider its implications. If we can identify a k-winner for any $k > 2$, then we know the existence and identity of the k'-winner (i.e., for any $k' > k$). The argument for Proposition 4 does not rely on the supposition that voters' ideal alternatives are uniformly distributed.

Proposition 4. *If alternative a is the k-winner for some $k > 2$, then alternative a is also the k'-winner for every $k' > k$.*

Proof. We start by supposing that the alternative a is the k-winner for some $k > 2$; thus, it is sufficient to show that a is the $k+1$-winner.

If a is not the $k+1$-winner, then some $a' \neq a$ is the winner on some collection, say K' of $k+1$ alternatives that include a. Therefore, we can select some $a'' \neq a'$ in K' such that a' is the winner on $K'' \equiv K' \backslash \{a''\}$: where

$$a'' \in \begin{cases} \{a_{\underline{K'}}, a_{\overline{K'}}\} & \text{if } a \text{ is interior to } K' \\ \{a_{\overline{K'}-1}, a_{\overline{K'}}\} & \text{if } a = a_{\underline{K'}} \\ \{a_{\underline{K'}}, a_{\underline{K'}+1}\} & \text{if } a = a_{\overline{K'}} \end{cases}$$

As $|K''| = k$, a cannot be the k-winner, thereby contradicting the premise. □

In our model, Propositions 3 and 4 imply that the 2-winner is the k-winner for every $k > 2$ provided they are either (i) exterior to N or (ii) also the 3-winner. More importantly, Proposition 4 implies that a 3-winner guarantees that this alternative is the k-winner for all k, potentially excluding $k = 2$. We can compare the result in Proposition 4 with a sequence of propositions from Meyers et al. (2014), proving the existence of the following three results:

1. A "profile" exists with no k-winner when $2 \leqslant k \leqslant k'$ but a k-winner exists for any $k' < k \leqslant n$.
2. A "profile" exists with no k-winner for any $k' < k \leqslant n$; however, there is a k-winner when $2 \leqslant k \leqslant k'$.
3. A "profile" exists where an alternative (say a) is a k-winner for any $2 \leqslant k \leqslant k'$, and a different alternative (say a') is the k-winner when $k' < k \leqslant n$.

They constructed profiles where k-winners come in "blocks". That is, a $2 \leqslant k' \leqslant n$ exists for which an alternative is the k-winner for (i) $k > k'$ or (ii) $k < k'$. In our model, $k' = 3$. If an alternative is the 3-winner, it is also the k-winner for every $3 \leqslant k \leqslant n$; however, this alternative might not be the 2-winner. In Section 4, Propositions 6 through 11 provide necessary and sufficient conditions for the existence (and non-existence) of a k-winner.

4. Results

This section is divided into three subsections. The first exclusively focuses on conditions under which the median alternative (interior to N) is the k-winner for every k. The second subsection presents more general results where any nonmedian (and interior to N) alternative $a_i \in K$ is the k-winner. The last subsection examines the special cases of the alternatives exterior to N.

4.1. Cases Where the Median Alternative Is the k-Winner

Consider the following condition:

$$\exists a_i \in N : \theta(a_i; N) > \max\left\{\sum_{j<i} \theta(a_j; N), \sum_{j>i} \theta(a_j; N)\right\} \quad (1)$$

The following proposition presents a sufficient (but not necessary) condition under which the median alternative (i.e., a_m) is the k-winner for all $k > 1$.

As shown in Figure 1, Condition (1) describes a collection of alternatives where a specific alternative has the largest vote share compared to the sum of the vote share of all alternatives to either side.

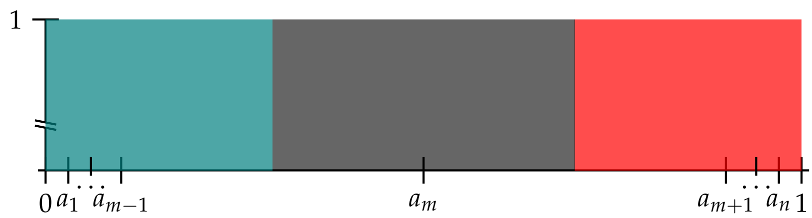

Figure 1. Graphical example of Condition (1).

Proposition 5. *If* (1) *holds, the median alternative, a_m, is the k-winner for all $k > 1$.*

Proof. We fix a k, and fix a $K \subseteq N$ containing a_m. Then, (1) implies that alternative a_m has the largest vote share (whenever $K = N$) and will beat any combination of $k - 1$ alternatives from the left or right. □

Propositions 6 through 9 will provide necessary and sufficient conditions for the existence (and nonexistence) of a k-winner for four specific cases, as outlined in Table 1.

Table 1. Summary of the four set of conditions on k and m, and associated propositions.

Case #	Condition	Used in
Case 1	$k \leqslant \min\{m, n - m\}$	Proposition 6
Case 2	$m \leqslant \min\{k, n - k\}$	Proposition 7
Case 3	$m \geqslant \max\{k, n - k\}$	Proposition 8
Case 4	$k \geqslant \max\{m, n - m\}$	Proposition 9

In each case, for each proposition, we have a (necessary) condition on both k and m. This set of inequalities defines different thresholds for either (i) the cardinality of K (i.e., k) or (ii) the ordinal position of the median alternative (i.e., m). In the first case, we need Lemmas 1 and 2 to prove Proposition 6, which provides necessary conditions under which a median alternative (i.e., a_m) is the k winner for every $k > 2$.

Lemma 1. *If $k \leqslant m$, then no alternative a_i for $a_i < a_m$ is a k-winner.*

Proof. Consider any collection K that contains a_m and $k - 1$ other alternatives, including a_i, all of which are less than a_m. Then, a_m is the winner on K; thus, a_i cannot be a k-winner. □

The Lemma 1 argument is as follows: Provided k is at most equal to the number of alternatives to the left of a_m (including m), we find a subset K composed of exactly these k alternatives. Then, the median alternative has the largest proportion of voters (to the left). For Lemma 2, the argument is analogous. Moreover, the number $n - m$ represents how many alternatives (including m) exist to the right of the median alternative.

Lemma 2. *If $k \leqslant n - m$, then no alternative a_i for $a_i > a_m$ is a k-winner.*

Proof. Consider any collection K that contains a_m and $k - 1$ other alternatives, including a_i, all of which exceed a_m. Then, a_m is the winner on K; thus, a_i cannot be a k-winner □

Next, we present sufficient and necessary conditions under which the median alternative is the k-winner for any given k.

Proposition 6. *Let $k \leqslant \min\{m, n - m\}$. For a given $k > 2$, the median alternative a_m is the k-winner if, and only if,*

$$\frac{1}{2}(a_m + a_{m-1}) < \frac{1}{2}(a_{m+1} - a_{m-1}) \text{ and}$$
$$1 - \frac{1}{2}(a_{m+1} + a_m) < \frac{1}{2}(a_{m+1} - a_{m-1})$$

Otherwise, there is no k-winner.

Proof. If $k \leqslant \min\{m, n - m\}$, no alternative a_i different than a_m can be a k-winner, as established in Lemmas 1 and 2. We fix any $K \in N$ such that $\{a_{m-1}, a_m, a_{m+1}\} \in K$. We must next construct the most damaging collection for alternative a_m, which would minimize a_m's vote. This collection would necessarily include neighbors a_{m-1} and a_{m+1}, yielding alternative a_m a share of $\theta(a_m; K) = \frac{1}{2}(a_{m+1} - a_{m-1})$. Now, suppose the following two conditions hold:

$$\frac{1}{2}(a_m + a_{m-1}) > \frac{1}{2}(a_{m+1} - a_{m-1}) \text{ and}$$
$$1 - \frac{1}{2}(a_{m+1} + a_m) > \frac{1}{2}(a_{m+1} - a_{m-1})$$

The first inequality implies that the proportion of voters that top-rank alternative a_{m-1} is strictly greater than the proportion of voters that top-rank alternative a_m if the subset of alternatives were $K = \{a_{m-1}, a_m, \ldots\}$. Then, the alternative a_m cannot be a k-winner. Nonetheless, since $k \leqslant m \leqslant n - k$ also holds, there cannot be a k-winner since the only potential alternative to be a k-winner was alternative a_m. Taking the second inequality, we have the same argument as before but for alternative a_{m+1}, which implies that there cannot be a k-winner (given that alternative a_m fails to be a k-winner). If neither of these conditions holds, then a_m is the k-winner. To show this, take any K that includes a_{m-1}, a_m and a_{m+1} and any $a_j \in K$ such that $j \leqslant m + 1$. Removing alternatives a_j for any $j < m - 1$ and replacing them with some alternatives a_j for any $j > m + 1$ increases the vote share of some remaining alternatives a_j for any $j < m - 1$ without increasing the vote share for alternative a_j for any $j > m$. However, the increased vote shares of a_j for any $j < m$ are bounded above by $\frac{1}{2}(a_m - a_{m-1})$; and alternative a_m then wins against a_{m-1}. The same argument applies to removing any alternative a_j for any $j < m - 1$. Finally, removing alternative a_{m-1} increases a_m's vote share and cannot increase the vote share of alternatives a_j for any $j < m - 1$ above $\frac{1}{2}(a_m - a_{m-1})$. By supposition, this share is less than $\frac{1}{2}(a_{m+1} - a_{m-1})$, which itself is less than a_m's new vote share. An equivalent argument prevents eliminating the alternative $a_m + 1$. □

Regarding the conditions in the last proposition, if both conditions hold, then the alternative a_m is the k-winner for a given k. Figure 2 shows the graphical interpretation of the following set of conditions:

$$\frac{1}{2}(a_m + a_{m-1}) < \frac{1}{2}(a_{m+1} - a_{m-1}) \text{ and}$$
$$1 - \frac{1}{2}(a_{m+1} + a_m) < \frac{1}{2}(a_{m+1} - a_{m-1})$$

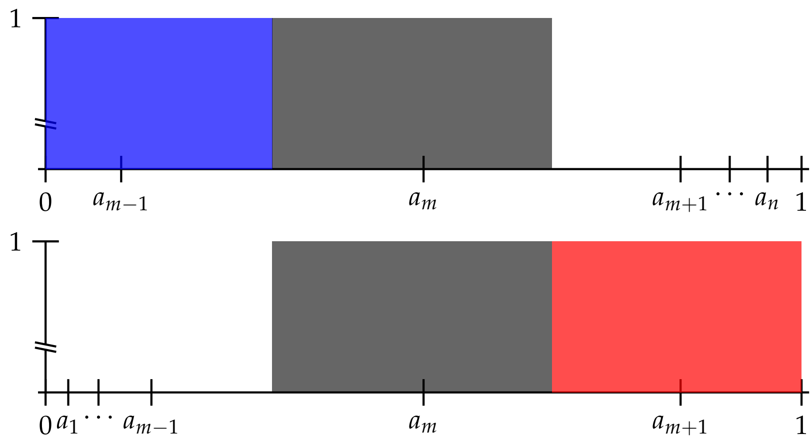

Figure 2. Graphical example of Proposition 6.

Additionally, by Proposition 4, if the median alternative is a k-winner for a given k, then it is also the k-winner for every $k > 2$. Therefore, the same conditions in Proposition 6 are sufficient and necessary for the median alternative, a_m, to be a k-winner for every $k > 2$. The next corollary formalises this result.

Corollary 1. *Let $k \leqslant \min\{m, n-m\}$. The median alternative a_m (which is interior to N) is the k-winner for every $k > 2$ if, and only if, $\frac{1}{2}(a_m + a_{m-1}) < \frac{1}{2}(a_{m+1} - a_{m-1})$ and $1 - \frac{1}{2}(a_{m+1} + a_m) < \frac{1}{2}(a_{m+1} - a_{m-1})$. Otherwise, there is no k-winner.*

4.2. Cases Where a Nonmedian Alternative Is the k-Winner

This subsection focuses on the results involving nonmedian alternatives. The next proposition involves the following sufficient condition: $m \leqslant \min\{k, n-k\}$. That is, (i) the size of the subset K and (ii) the number of alternatives excluded from K are weakly larger than the median alternative's ordinal position.

Proposition 7. *Let $m \leqslant \min\{k, n-k\}$. For any $a_1 < a_i < a_m$ and for any given k alternative, a_i is the k-winner if, and only if,*

$$a_{i-1} > \frac{1}{2}(a_{i+1} - a_i) \text{ and } a_{i+1} - a_{i-1} > \max\{a_{n-k+i+1} - a_i, 2 - a_n - a_{k-1}\}.$$

Otherwise, there is no k-winner.

Proof. First, Lemma 2 shows that no alternative $a_i > a_m$ can be the k-winner. We now take any $a_i < a_m$. As before, we must construct the most damaging collection of alternatives for alternative a_i. Three possible contenders to beat a_i include a_{i-1}, a_{i+1} and a_n. In each case, the contender has the best chance of beating a_i when K contains a_{i-1} and a_{i+1} (by minimizing a_i's vote share), in which case, a_i's share is $\frac{1}{2}(a_{i+1} - a_{i-1})$.

Alternative $a_i - 1$'s vote share is maximized when K contains no alternative a_k for any $j < i-1$. Subsequently, $a_i - 1$ then beats a_i if, and only if,

$$\frac{1}{2}(a_i + a_{i-1}) > \frac{1}{2}(a_{i+1} - a_{i-1}) \implies a_{i-1} > \frac{1}{2}(a_{i+1} - a_i)$$

Alternative a_{i+1}'s vote share is maximized when $K = \{a_1, \ldots, a_{i+1}, a_{n-k+i+1}, \ldots, a_n\}$; and a_{i+1} then beats a_i if, and only if, $a_{n-k+i+1} - a_i > a_{i+1} - a_{i-1}$. Finally, a_n's vote share is maximized when $K = \{a_1, \ldots, a_{k-1}, a_n\}$. a_n then beats a_i if, and only if, $2 - a_n - a_{k-1} > a_{i+1} - a_{i-1}$. Therefore, a_i cannot be the k-winner if one of these conditions holds.

Conversely, no alternative $a_j < a_{i-1}$ can beat a_i if a_{i-1} cannot beat a_i because alternative a_i's vote share exceeds the maximum share for alternative a_{i-1} (the upper bound for any alternative a_j for any $j < i-1$).

Similarly, no alternative a_j for any $j > i$ (other than a_{i+1} and a_n) can beat a_i if neither a_{i+1} nor a_n can beat a_i. □

Unlike the previous proposition, the next result deals with the following condition: $m \geqslant \max\{k, n-k\}$. That is, (i) the size of the subset K and (ii) the number of alternatives excluded from K are weakly smaller than the median alternative's ordinal position.

Proposition 8. *Let $m \geqslant \max\{k, n-k\}$. For any $a_m < a_i < a_n$ and for any given k, the alternative a_i is the k-winner if, and only if,*

$$\frac{1}{2}(a_i - a_{i-1}) > 1 - a_{i+1}, \text{ and } a_{i+1} - a_{i-1} > \max\{a_{i-1} - a_{i-n+k-2}, a_1 + a_{n-k+2}\}.$$

Otherwise, there is no k-winner.

Proof. No alternative $a_i < a_m$ can be the k-winner, which Lemma 1 implies. Now, take any $a_i > a_m$. The most damaging collection of contending alternatives for a_i must include a_1, a_{i-1}, and a_{i+1}. By including a_{i-1} and a_{i+1} in K, we maximize the contenders' vote share and minimize a_i's vote share. Then, a_i's share would be $\frac{1}{2}(a_{i+1} - a_{i-1})$. Alternative a_{i+1}'s vote share is maximized when K contains no alternative $a_j > a_{i+1}$. In this case, a_{i+1}, then beats a_i if, and only if,

$$1 - \frac{1}{2}(a_{i+1} - a_i) > \frac{1}{2}(a_{i+1} - a_{i-1}) \implies 1 - a_{i+1} > \frac{1}{2}(a_i - a_{i-1}).$$

Alternative a_{i-1}'s vote share is maximized when $K = \{a_1, \ldots, a_{i-n+k-2}, a_{i-1}, a_i, \ldots\}$; thus, alternative a_{i-1} then beats alternative a_i if, and only if, $a_{i-1} - a_{i-n+k-2} > a_{i+1} - a_{i-1}$. Finally, alternative a_1's vote share is maximized when $K = \{a_1, a_{n-k+2}, \ldots, a_{i-1}, \ldots\}$; thus, a_1 then beats a_i if, and only if, $a_1 + a_{n-k+2} > a_{i+1} - a_{i-1}$. Therefore, a_i cannot be the k-winner if one of these conditions holds.

Conversely, no alternative $a_j > a_{i+1}$ can beat a_i if a_{i+1} cannot beat a_i. Alternative a_i's vote share is strictly greater than the maximum share for alternative a_{i+1}, which is the upper bound for any alternative $a_j > a_{i+1}$. Similarly, no alternative $a_j < a_i$ (other than a_{i-1} and a_1) can beat a_i if neither a_{i-1} nor 1 can beat a_i. □

The final result of this subsection examines the last case, which is the converse of the conditions from the first case (i.e., Lemmas 1 and 2). This proposition constructs sufficient and necessary conditions under which any alternative a_i is the k-winner.

Proposition 9. *Let $k \geqslant \max\{m, n-m\}$. For any $a_1 < a_i < a_n$ and for any given k, the alternative a_i is the k-winner if, and only if,*

$$a_{i+1} - a_{i-1} > \max\left\{a_1 + a_{n-k+2}, a_i - a_{i-n+k-2}, a_{n-k+i+1} - a_{i+1}, 2 - a_n - a_{k-1}\right\}$$

Proof. The most damaging collection of contending alternatives for any a_i must include t a_1, a_{i-1}, a_{i+1}, and a_n. By including a_{i-1} and a_{i+1} in a_K, we maximize the contenders' vote share and minimize a_i's vote share. Then, the alternative a_i's share would be $\frac{1}{2}(a_{i+1} - a_{i-1})$. First, alternative 1's vote share is maximized when we push all $k-1$ alternatives to the right, i.e., $K = \{a_1, a_{n-k+2}, \ldots, \}$. Therefore, a_1 then beats a_i if, and only if, $a_1 + a_{n-k+2} > a_{i+1} - a_{i-1}$. Then, alternative $a_i - 1$'s vote share is maximized when $K = \{a_1, \ldots, a_{i-n+k-2}, a_{i-1}, \ldots\}$; thus, $a_i - 1$ then beats a_i if, and only if, $a_{i-1} - a_{i-n+k-2} > a_{i+1} - a_{i-1}$. Alternative a_{i+1}'s vote share is maximized when $K = \{a_1, \ldots, a_{i+1}, a_{n-k+i+1}, \ldots, a_n\}$; thus, $i+1$ then beats i if, and only if, $a_{n-k+i+1} - a_{i+1} > a_{i+1} - a_{i-1}$. Finally, alternative a_n's vote share is maximized when $K = \{a_1, \ldots, a_{k-1}, a_n\}$; thus a_n then beats a_i if, and only if, $2 - a_n - a_{k-1} > a_{i+1} - a_{i-1}$. Consequently, a_i cannot be the k-winner if one of these conditions holds.

Conversely, either $a_1 + a_{n-k+2}$ or $a_i - a_{i-n+k-2}$ (the vote share of the contenders 1 and $i-1$) constitute the upper bound for alternative j's vote share for any $j > i$. The condition implies that alternative i beats alternatives 1 and $i-1$; thus, only the alternative a_i can be the k-winner. Similarly, $a_{n-k+i+1} - a_{i+1}$ or $2 - a_n - a_{k-1}$ represents the upper bound for the vote share of any alternative to the left of a_i. The condition implies that alternative a_i beats both alternatives a_{i+1} and a_n; therefore, only a_i must be the k-winner. □

4.3. Alternatives Exterior to N

This subsection presents a special case: alternatives exterior to N. The following two propositions provide sufficient and necessary conditions under which the two alternatives exterior to N (i.e., a_1 and a_n) are the k-winners for any given k. Given their spatial location, alternatives exterior to N must be considered differently. Notably, calculating the vote share of the alternatives exterior to N depends on the identity of the neighboring alternative. For example, the vote share for a_1 would differ if a_2 is present instead of a_3; therefore, we use similar strategies to prove the following two propositions. In Proposition 10, we fix a subset of k alternatives, including alternative a_1. We then examine the most damaging collection of alternatives for alternative a_1. We must consider every possible case and explore every alternative that could lie next to alternative a_1. We then derive the conditions under which the alternative a_1 would beat the remaining alternatives in every case. Finally, we generalize this sequence of conditions. The resulting set of conditions is sufficient and necessary for alternative a_1 to be the k-winner for every $k > 2$.

Proposition 10. *Alternative a_1 is the k-winner if* $\forall a_1 < a_j \in N, \forall a_\ell < a_j$:

$$a_1 + a_2 > \max_{1 \leqslant \ell \leqslant k-2} \left\{ a_{n-k+\ell+2} - a_\ell, 2 - a_{k-1} - a_k \right\} \tag{2}$$

Proof. See Appendix B. □

In the proof of Proposition 11, we follow the same case-by-case approach we used in Proposition 10. More specifically, we present sufficient and necessary conditions for alternative a_n to be the k-winner for every $k > 2$.

Proposition 11. *Alternative a_n is the k-winner if* $\forall a_j < a_n, \forall \ell > j$:

$$2 - a_n - a_{n-1} > \max_{2 \leqslant \ell \leqslant k-1} \left\{ a_{n-k+2} + a_{n-k+3}, a_{n-k+\ell+2} - a_{\ell-1}, a_n - a_{k-1} \right\} \tag{3}$$

Proof. See Appendix C. □

5. Examples

This section presents four examples that illustrate the following cases:

1. An alternative exterior to N may be the k-winner for every k.
2. The 2-winner and the 3-winner both exist but differ.
3. The 2-winner and the n-winner coincide; however, no k-winner exists for every $2 < k < n$.
4. The 2-winner and the n-winner differ; however, no k-winner exists for every $2 < k < n$.

The interval for every graph is the unit interval. The black marks represent the position of the alternatives. The blue-shaded area represents the vote share of the 2-winner; the red-shaded area represents the vote share of the n-winner. In the first and third examples, the 2-winner and the n-winner are the same alternative; in this case, there is only one shaded area. The main difference between the first and last two cases is that in the latter, a k exists for which there is no k-winner.

5.1. First Case

Consider the parametric configuration in Figure 3. In this first case, we have a graphical example of Proposition 2. One of the alternatives' vote share is strictly greater than half. Alternative 3's vote share is 0.505; thus, alternative 3 is the k-winner for every k. Moreover, this scenario is a graphical example of Proposition 3 since alternative 3 is an alternative exterior to N.

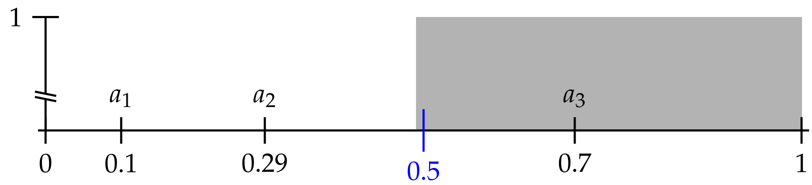

Figure 3. An alternative exterior to N is k-winner for every k.

5.2. Second Case

Consider the parametric configuration in Figure 4. The 2-winner and 3-winner now exist; however, they are not the same alternative. Alternative 2 is the 2-winner, whereas alternative 3 is the 3-winner. The position of the alternatives in this case is very similar to the previous one; however, in this case, alternative 2's position is closer to the median voter's ideal, close enough to become the median voter's ideal and the median alternative.

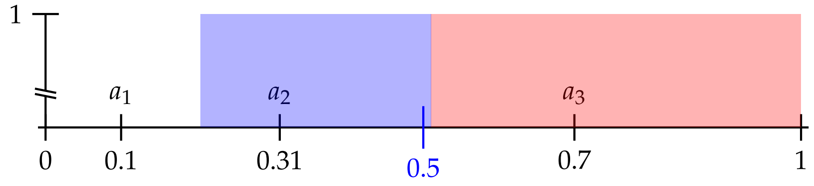

Figure 4. The 2-winner and the 3-winner both exist but differ.

5.3. Third Case

Consider the parametric configuration in Figure 5. In this particular case of the four alternatives, the 2-winner and n-winner (i.e., 4-winner) are the same alternatives; however, the 3-winner does not exist.

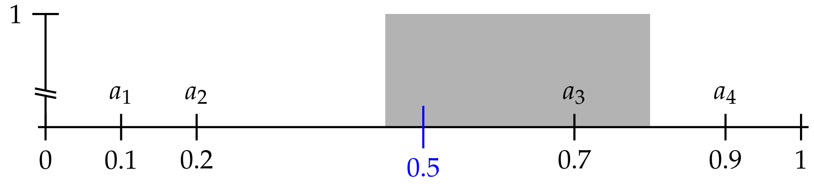

Figure 5. The 2-winner and the n-winner coincide but no k-winner exists for every $2 < k < n$.

5.4. Fourth Case

Consider the parametric configuration in Figure 6. Also, in this case, we have that $n = 4$. In this case, the 2-winner and 4-winner exist but are not the same alternative; however, the 3-winner does not exist.

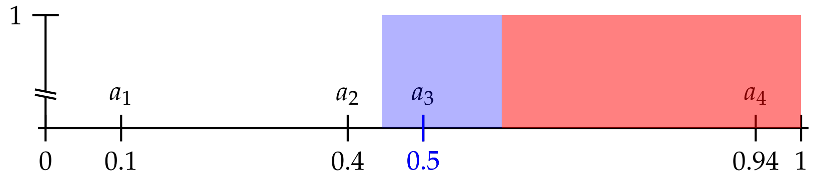

Figure 6. The 2-winner and the n-winner differ; however, no k-winner exists for every $2 < k < n$.

We next discuss the link between Proposition 4 and the parametric configuration in both the Third and Fourth cases. Proposition 4 does not hold because of the necessary condition of existence of the k-winner for $k > 2$. Conversely, in the First case, Proposition 4 trivially holds because only three alternatives are available.

6. Conclusions

In 1785, the Marquis of Condorcet conceptualized a simple mechanism to identify the electoral winner among many alternatives: look at the winner of every pair. This criterion would remain dormant for centuries until, in the 1950s, the Scottish economist Duncan Black would resurrect the Marquis concept and derive another simple but potent concept: single-peakedness. Since Black, the idea of a Condorcet winner has become central to many models in political economy (e.g., median voter theorem). This paper used a spatial approach to present a spatial extension of the Condorcet winner: a k-winner. Instead of looking at pairs of alternatives, we look at subsets composed of k alternatives in our model. The main contribution is twofold. (i) We present sufficient and necessary conditions for any alternative to be the k-winner for any $k \geqslant 2$. (ii) We also show that if a k-winner exists for any $k > 2$, then this alternative is also the k'-winner for all $k < k' < n$. The results of this paper can be extended in several directions. Proposition 4 can be translated into a testable prediction; however, econometric challenges arise since the polling and electoral data cannot contain all the respondents' ranked preferences because most polls only ask for top-rank preferences.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article, further inquiries can be directed to the corresponding author.

Conflicts of Interest: The author declares no conflicts of interest.

Appendix A. Non-Uniformly Distributed Ideals

Let the voter's ideal alternatives follow a distribution f, with support on $[0,1]$, which we assume is continuous and has a unique median m and cumulative distribution function F. Now, we fix a subset $K \subseteq N$ with cardinality equal to k, such that the alternatives $\{a_{i^-}, a_i, a_{i^+}\} \subseteq K$ and let a_{i^-}, and a_{i^+} are the respective left and right "neighbor" alternatives for a_i, such that

$$a_{i^-} = \max\{a_j \in K : a_j < a_i\} \text{ and } a_{i^+} = \min\{a_j \in K : a_j > a_i\}.$$

Given a $K \in N$, we re-define the proportion of voters whose ideal alternative is $a_i \in K$ as follows:

$$\theta(a_i;K) = \begin{cases} F\left(\dfrac{a_i + a_{i^+}}{2}\right) & \text{if } a_i = \min\{a_j \in K\} \\ 1 - F\left(\dfrac{a_i + a_{i^-}}{2}\right) & \text{if } a_i = \max\{a_j \in K\} \\ F\left(\dfrac{a_i + a_{i^+}}{2}\right) - F\left(\dfrac{a_i + a_{i^-}}{2}\right) & \text{otherwise} \end{cases}$$

Proposition A1. *Let $k \leqslant \min\{m, n-m\}$. For a given $k > 2$, the median alternative a_m (which is interior to N) is the k-winner if, and only if,*

$$\forall K \in N \text{ such that } \{a_{m-1}, a_m, a_{m+1}\} \in K : \begin{cases} F\left(\dfrac{a_m + a_{m-1}}{2}\right) < \theta(a_m;K) \\ 1 - F\left(\dfrac{a_m + a_{m+1}}{2}\right) < \theta(a_m;K) \end{cases}$$

Otherwise, there is no k-winner.

Proof. If $k \leqslant \min\{m, n-m\}$, then no alternative a_i other than a_m can be a k-winner, as established in Lemmas 1 and 2. We must construct the most damaging collection for alternative a_m, which would minimize a_m's vote. This collection would necessarily include neighbors a_{m-1} and a_{m+1}, yielding alternative a_m a share of $\frac{1}{2}(a_{m+1} - a_{m-1})$. Suppose the following two conditions hold:

$$F\left(\frac{a_m + a_{m-1}}{2}\right) > \theta(a_m;K) \text{ and}$$
$$1 - F\left(\frac{a_m + a_{m+1}}{2}\right) > \theta(a_m;K)$$

The first condition, which implies that the proportion of voters that top-rank alternative a_{m-1} is strictly greater than the proportion of voters that top-rank alternative a_m if the subset of alternatives were $K = \{a_{m-1}, a_m, \ldots\}$. Then, the alternative a_m cannot be a k-winner. Nonetheless, since $k \leqslant m \leqslant n-k$ also holds, there cannot be a k-winner since the only potential alternative to be a k-winner was alternative a_m. Taking the second condition, we have the same argument as before but for alternative a_{m+1}, which would also imply that there cannot be a k-winner (given that alternative a_m fails to be a k-winner). If neither condition holds, then a_m is the k-winner. To show this, take any K that includes a_{m-1}, a_m and a_{m+1} and any $a_j \in K$ such that $j \leqslant m+1$. If we remove alternatives a_j for any $j < m-1$ and replace them with some alternatives a_j for any $j > m+1$, the vote share of some remaining alternatives a_j increases for any $j < m-1$ without raising the vote share for alternative a_j for any $j > m$. In contrast, the increased vote shares of a_j for any $j < m$ are bounded above by $F\left(\frac{a_m + a_{m-1}}{2}\right)$; alternative a_m then wins against a_{m-1}. The same argument applies to removing any alternative a_j for any $j < m-1$. Finally, removing alternative a_{m-1} increases a_m's vote share and cannot increase the vote share of alternatives a_j for any $j < m-1$ above $1 - F\left(\frac{a_m + a_{m+1}}{2}\right)$. By supposition, this share is less than $\theta(a_m; K)$, which itself is less than a_m's new increased vote share (given alternative a_{m-1}'s removal). An equivalent argument precludes eliminating the alternative a_{m+1}. □

Appendix B. Proof of Proposition 10

Proof. First, fix $K = \{a_1, a_{i_1}, a_{i_2}, \ldots, a_{i_{k-1}}\}$. Given this subset, we compute the maximum vote share for every alternative other than a_i. We then examine every case where a_j could take the other possible locations in $K \setminus \{a_1\}$.

1. Case $j \equiv i_1$. For alternative 1 to beat alternative $j = i_1$, we need: $\theta(a_1; K) > \theta(a_{i-1}; K)$, which implies: $a_1 + a_{i_1} > a_{i_2} - a_1$. The farthest location for i_2 has to be $n-k+3$. Then:

$$a_1 + a_2 > a_{n-k+3} - a_1 \quad (\text{and } a_j > a_1) \tag{A1}$$

2. Case $j \equiv i_2$. Following the previous case, we must have $\theta(a_1; K) > \theta(a_{i_2}; K)$, which implies: $a_1 + a_{i_1} > a_{i_3} - a_{i_1}$. The farthest location for i_3 has to be $n-k+4$ and for i_1 is 2. Then:

$$a_1 + a_2 > a_{n-k+4} - a_2 \quad (\text{and } a_j > a_2) \tag{A2}$$

3. Case $i_3 < j \equiv i_\ell < i_{k-2}$ Again, for alternative 1 to beat alternative $j \equiv i_\ell$, it would be necessary and sufficient that $\theta(a_1; K) > \theta(a_{i_\ell}; K)$, which implies: $a_1 + a_{i_{i_1}} > a_{i_{\ell+1}} - a_{i_{\ell-1}}$. The farthest location for i_ℓ has to be $n-k+\ell+2$ and for i_ℓ is $\ell+1$. Then:

$$a_1 + a_2 > a_{n-k+\ell+2} - a_\ell \quad (\text{and } a_j > a_\ell) \tag{A3}$$

4. Case $j \equiv i_{k-2}$. Again, $\theta(a_1; K) > \theta(a_{i_{k-2}}; K) \implies a_1 + a_{i_{i_1}} > a_{i_{k-1}} - a_{i_{k-3}}$. Then, we must set i_{k-3} to be $k-2$ and i_{k-1} to be n. Then:

$$a_1 + a_2 > a_n - a_{k-2} \quad (\text{and } a_j > a_{k-2}) \tag{A4}$$

5. Case $j \equiv i_{k-1}$. Lastly, we have

$$\theta(a_1; K) > \theta(a_{i_{k-1}}; K) \implies a_1 + a_{i_1} > 2 - a_{i_{k-2}} - a_{i_{k-1}}.$$

Then, we must set i_{k-2} to be $k-1$ and i_{k-1} to be k. Then:

$$a_1 + a_2 > 2 - a_{k-1} - a_k \tag{A5}$$

□

Appendix C. Proof of Proposition 11

Proof. First, fix $K = \{a_{i_1}, \dots, a_{i_{k-1}}, a_n\}$. Alternative a_n will be the k-winner if, and only if, $\forall a_j < a_n : \theta(a_n; K) > \theta(a_j; K)$. We then examine every case where a_j could take the other possible locations.

1. Case $a_j \equiv a_{i_1}$. The first necessary and sufficient condition would be: $\theta(a_n; K) > \theta(a_{i_1}; K) \implies 2 - a_n - a_{n-1} > a_{i_2} + a_{i_1}$. Since i_1 must be $n - k + 2$ and i_2 must be $n - k + 3$. Then:

$$2 - a_n - a_{n-1} > a_{n-k+2} + a_{n-k+3} \tag{A6}$$

2. Case $a_j \equiv a_{i_2}$. Then, we must have:

$$\theta(a_n; K) > \theta(a_{i_2}; K) \implies 2 - a_n - a_{n-1} > a_{i_3} - a_{i_1}.$$

The farthest location for i_3 has to be $n - k + 4$ and for i_2 is 1. Then:

$$2 - a_n - a_{n-1} > a_{n-k+4} - a_1 \tag{A7}$$

3. Case $a_{i_2} < a_j \equiv a_{i_\ell} < a_{i_{k-1}}$. Again, we must have: $\theta(a_n; K) > \theta(a_{i_\ell}; K) \implies a_{n+1} - a_{n-1} > a_{i_{\ell+1}} - a_{i_{\ell-1}}$. The farthest location for $i_{\ell+1}$ has to be $n - k + \ell + 2$ and for $i_{\ell-1}$ is $\ell - 1$. Then:

$$2 - a_n - a_{n-1} > a_{n-k+\ell+2} - a_{\ell-1} \tag{A8}$$

4. Case $a_j \equiv a_{i_{k-1}}$. Lastly, we must have

$$\theta(a_n; K) > \theta(a_{i_{k-1}}; K) \implies 2 - a_n - a_{n-1} > a_n - a_{i_{k-2}}$$

Then, set i_{k-2} to be $k - 1$. Then:

$$2 - a_n - a_{n-1} > a_n - a_{k-1} \tag{A9}$$

□

Note

1 Generic collections are those with the property that if a_i and $a_{i'}$ are distinct elements of N, then $a_i + a_{i'} \neq 1$.

References

Black, D. (1958). *The theory of committees and elections*. Cambridge University Press.

Condorcet, M. d. (1785). *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*. Cambridge University Press.

Meyers, A., Orrison, M. E., Townsend, J., Wolff, S., & Wu, A. (2014). Generalized condorcet winners. *Social Choice and Welfare*, *43*(1), 11–27. [CrossRef]

Newing, R. A., & Black, D. (1951). *Committee decisions with complementary valuation*. William Hodge.

Saari, D. G. (1989). A dictionary for voting paradoxes. *Journal of Economic Theory*, *48*(2), 443–475. [CrossRef]